Appendix 2.1 Some Rules of Matrix Algebra

Many texts cover the basic elements of matrix (or linear) algebra (e.g., Campbell 1965; Finkbeiner 1966; Bradley 1975; Searle 1982). Only a few simple rules for matrix operations are reviewed here.

Addition of two column vectors. The addition of two column vectors is similar to the addition of ordinary numbers. Consider the two-trait case of equation 2.1, $\mathbf{z} = \mathbf{a} + \mathbf{e}$, which represents

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

The vector sum is

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a_1 + e_1 \\ a_2 + e_2 \end{bmatrix}.$$

or $z_1 = a_1 + e_1$ and $z_2 = a_2 + e_2$.

Addition of two matrices. Matrix addition is a simple extension of the rule for adding two vectors. Thus, the two-trait case of equation 2.2, $\mathbf{P} = \mathbf{G} + \mathbf{E}$, is

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} + \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix}$$

or

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = \begin{bmatrix} G_{11} + E_{11} & G_{12} + E_{12} \\ G_{12} + E_{12} & G_{22} + E_{22} \end{bmatrix}$$

Thus, the phenotypic variance for trait 1 (P_{11}) is the sum of the trait's additive genetic variance (G_{11}) and its environmental variance (E_{11}).

In general, the lower left-hand element in a 2 × 2 matrix would not be equal to the upper right-hand element and would be denoted P_{21}, G_{21}, or E_{21}. However, our **P**-, **G**- and **E**-matrices are *symmetric*, which means that $P_{12} = P_{21}$, $P_{13} = P_{31}$, etc.

Multiplying a matrix and a column vector. The basic rule to remember in multiplying a matrix and a vector or two matrices is "rows times columns." For example, consider the two-trait case of equation 2.4, $\mathbf{\Delta \bar{z}} = \mathbf{G}\boldsymbol{\beta}$, which represents

$$\begin{bmatrix} \Delta\bar{z}_1 \\ \Delta\bar{z}_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

To evaluate the product $\mathbf{G}\boldsymbol{\beta}$, we first multiply the first row of **G** by the column $\boldsymbol{\beta}$. The product is the first element in the first row of **G**, G_{11}, times the first element of $\boldsymbol{\beta}$, β_1, plus the second element of the first row of **G**, G_{12}, times the second element of $\boldsymbol{\beta}$, β_2. In other words,

$$\Delta\bar{z}_1 = G_{11}\beta_1 + G_{12}\beta_2.$$

Likewise, the second element of the product $\mathbf{G}\boldsymbol{\beta}$ is

$$\Delta\bar{z}_2 = G_{12}\beta_1 + G_{22}\beta_2.$$

The rule is easily extended to the many-trait case.

Multiplying two matrices. Again, the rule is "rows times columns," but now there is more than one column. Consider the second term in equation 2.11, $\frac{1}{2}\mathbf{C_{az}M}^{\mathrm{T}}$, in the two-trait case. Before we take the product $\mathbf{C_{az}M}^{\mathrm{T}}$, we note that $\mathbf{M}^{\mathrm{T}}$ is the so-called *transpose* of the matrix **M**; the rows of **M** are the columns of $\mathbf{M}^{\mathrm{T}}$. In other words, if

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

then

$$\mathbf{M}^{\mathrm{T}} = \begin{bmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{bmatrix},$$

The product $\mathbf{C_{az}M}^{\mathrm{T}}$ is then

$$\begin{bmatrix} C_{az_{11}} & C_{az_{12}} \\ C_{az_{21}} & C_{az_{22}} \end{bmatrix} \begin{bmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{bmatrix}$$

The upper left-hand element (with subscript 11) of the product is the *first* row of $\mathbf{C_{az}}$ times the *first* column of $\mathbf{M}^{\mathrm{T}}$; the upper right-hand element of the product (subscript 12) is the *first* row of $\mathbf{C_{az}}$ times the *second* column of $\mathbf{M}^{\mathrm{T}}$; the lower left-hand element (subscript 21) of the product is the *second* row of $\mathbf{C_{az}}$ times the *first* row of $\mathbf{M}^{\mathrm{T}}$, and finally, the lower right-hand element (22) of the product is the *second* row of $\mathbf{C_{az}}$ times the *second* row of $\mathbf{M}^{\mathrm{T}}$. Our matrix product is

$$\begin{bmatrix} C_{az_{11}}M_{11} + C_{az_{12}}M_{12} & C_{az_{11}}M_{21} + C_{az_{12}}M_{22} \\ C_{az_{21}}M_{11} + C_{az_{22}}M_{12} & C_{az_{21}}M_{21} + C_{az_{22}}M_{22} \end{bmatrix}$$

The symbol $\frac{1}{2}$ in front of $\mathbf{C_{az}M}^{\mathrm{T}}$ in equation 2.11 denotes an ordinary number or scalar. Multiplying the scalar $\frac{1}{2}$ times the product $\mathbf{C_{az}M}^{\mathrm{T}}$ means that each element in $\mathbf{C_{az}M}^{\mathrm{T}}$ is to be multiplied by $\frac{1}{2}$. Thus,

$$\frac{1}{2}\mathbf{C_{az}M}^{\mathrm{T}} = \begin{bmatrix} \frac{1}{2}C_{az_{11}}M_{11} + \frac{1}{2}C_{az_{12}}M_{12} & \frac{1}{2}C_{az_{11}}M_{21} + \frac{1}{2}C_{az_{12}}M_{22} \\ \\ \frac{1}{2}C_{az_{21}}M_{11} + \frac{1}{2}C_{az_{22}}M_{12} & \frac{1}{2}C_{az_{21}}M_{21} + \frac{1}{2}C_{az_{22}}M_{22} \end{bmatrix}$$

Matrix multiplication is unlike ordinary or scalar multiplication in that the order of multiplication matters. Notice that $\mathbf{C_{az}M}^{\mathrm{T}} \neq \mathbf{M}^{\mathrm{T}}\mathbf{C_{az}}$!